# Understanding Lobachevski

Not quite to infinity and not quite beyond.

By Mark Lawton
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One of the most intriguing aspects of Lobachevski's Theory of Parallels is the manner in which he repeats processes over and over again until the conclusion he desires becomes apparent. We see this in theorem 20 where he scales a grid until it is sufficiently large, in theorem 21 where an angle is decreased until sufficiently small, and in 23 where a triangle is increased until the sum of its angles becomes negative. The process is first introduced in theorem 19 and its use there (I conclude after a dozen hours and a multitude of attempts) is particularly problematic. Theorem 19 states: *"In a rectilineal triangle the sum of the angles can not be greater than two right angles."* [1] The technique of the proof is as follows:

1) Find the shortest side of the given triangle (figure 1, side BC)
2) Bisect it at D.
3) Prolong AD to E such that AD = DE.
4) Create a new triangle as indicated in the figure.
5) Repeat steps 1, 2, & 3 indefinitely

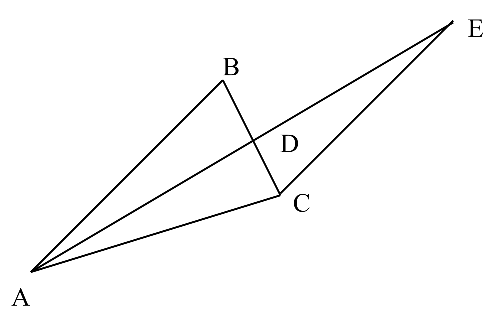


Figure 1

The argument is that by following the procedure one will end up with angle CAE & angle AEC equivalent. As they approach equality, angle ACE approaches *P*. Since the sum of the angles must equal $P + a$, the two (very) acute angles each must each be less than or

[1] Lobachevski, Theory of Parallels, Page 16

equal in magnitude to one-half of *a*. If the process is repeated further, they drop below one-half of *a* and we see that the total sum of the angles can not be greater than *P*.

Before grappling with the final steps of the proof, let's understand why it should work. The proof hinges on the relationship of the three sides of the triangle as the process is repeated. For ease of understanding assume AB is the longest side of Professor if this note is still here then your student plagiarized my paper from pnca.edu slash tilda mlawton triangle ABC, AC is the medium side, and BC is the shortest. By bisecting the shortest side, Lobachevski insures that the triangle grows in the following manner.

The new side (CE) will always be longer than the previous medium side (AC):

| | | |
|---|---|---|
| | Proof: | |
| I | AD = DE | constructed as such |
| II | BD = DC | definition of bisection |
| III | angle ADB = angle EDC | vertical angles, theorem 6 |
| IV | triangle ADB = triangle EDC | SAS, theorem 10 |
| V | CE = AB | corresponding sides of congruent triangles |
| VI | AB > AC | AB is identified initially as the longest side |
| VII | CE > AC | see previous two steps |

The long side (AB) will always get longer (AE):

| | | |
|---|---|---|
| | Proof: | |
| I | AB > BC | AB is identified initially as the longest side |
| II | angle ACB > angle BAC | big sides have big angles, theorem 9 |
| III | angle BAC > angle DAC | the whole is greater than the part |
| IV | angle ACB much > angle DAC | see previous two steps |
| V | angle ACE > angle ACB | the whole is greater than the part |
| VI | angle ACE much > angle DAC | see previous two steps |
| VII | AE > CE | big angles have big sides, theorem 9 |
| VII | CE = AB | see previous proof, corresponding sides congruent triangles |
| VIII | AE > AB | see previous two steps |

Since the long side always gets longer we will never bisect it. The short side however always becomes the new medium side and the medium side always becomes the new short side. That is, they always switch roles. We will always therefore switch back and forth between bisecting the left side of the triangle and bisecting the right side. Each time through the process the triangle grows. First the right side gets longer then the left and so on forever.

The question then becomes how do we know that sides AC and CE will eventually be equal. (Recall, that we need them to be equal to insure that we end up with an isosceles triangle and therefore the corresponding angles will be equal.) One method to prove this is to assume the extreme opposite. When angle ACE is just about *P* let's make AC be very long (let's call the length x in figure 2) and CE almost zero. Since CE is almost zero, AE will also equal x. When we repeat the process, the new length, CE' will equal x and we will have the desired isosceles triangle. Q.E.D. – Not quite.

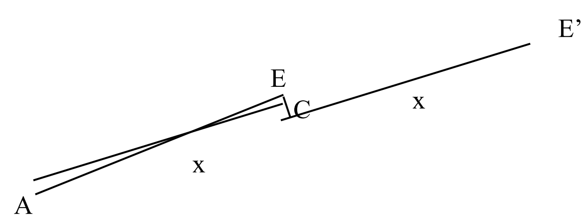


Figure 2

The problem arises if we repeat the process one more time. Starting with the isosceles triangle with legs x and x we choose to bisect the right one. After all, we know that the triangle isn't quite yet isosceles. When we follow the given steps the new lengths will be x and 2x (a ratio of 2:1). If we do it again the lengths will be 2x and 3x (a ratio of 1.5:1). The process will continue as such:

| Left side of triangle | Right side | Ratio | |
|---|---|---|---|
| x | x | 1 : 1 | fl Assumed |
| x | 2x | 2 : 1 | |
| 3x | 2x | 1.5 : 1 | |
| 3x | 5x | 1.67 : 1 | |
| 8x | 5x | 1.6 : 1 | |
| 8x | 13x | 1.625 : 1 | |

We see that the process is not stable; if the sides have a ratio of 1:1 they will not preserve it. In fact, it isn't clear how we arrive at the desired isosceles triangle in the first place!

Using the law of sines in conjunction with the law of cosines we can simulate the effects of repeating this process and the results are shown on the enclosed spreadsheet. The key features are circled and show that even as the large angle approaches a straight line the corresponding sides are not equal. (We do know that the base angles approach zero but that doesn't mean that they are equal.)

The fact that we can not show that the process yields an isosceles triangle leads us to believe that Lobachevski has a different approach. The obvious candidate is to demonstrate that the angles themselves approach equality without reliance on the sides. The strategy would be as follows:

I Sum of angles = *P* + a

II Repeat process as follows:

| Angle EAC | Angle ACE | Angle CEA |
|---|---|---|
| .8a | < Pi | .2a |
| .3a | < Pi | .7a |
| .6a | < Pi | .4a |
| .55a | < Pi | .45a |
| .5a | < Pi | .5a |

| | | |
|---|---|---|
| <.5a | Pi | <.5a |

The reason the two base angles alternate as bigger and smaller (columns 1 & 3 in the table) is that their corresponding sides alternate in size (proof given above). This strategy is compelling but certainly not a proof. Especially problematic is the fact that columns 1 & 3 in the table must add to a number greater than or equal to *a* but the method itself makes both of them always get smaller (see table below).

I Initially sum of angles = $P$ + a, but both always get smaller

II Repeat process as follows:

| Angle EAC | Angle ACE | Angle CEA |
|---|---|---|
| .8a | < Pi | .2a |
| .2a | < Pi | .6a |
| .4a | < Pi | .2a |
| .1a | < Pi | .3a |
| .2a | < Pi | .1a |
| .06a | < Pi | .04a |

Lobachevski's proof seems to rely on the sum of the small angles equaling *a* and, simultaneously, the angles getting smaller! This is quite perplexing. I conclude that this requires more thought.